Physics 618 2020

March 27, 2020

Central Extensions

Can have same $\tilde{G}$ and
and non-iso-
exts.

$$1 \to A \xrightarrow{\iota} \tilde{G} \xrightarrow{\pi} G \to 1$$

$$\iota(A) \subset Z(\tilde{G})$$

Thm: $\bar{\mathcal{E}}(G, A)$ set of isom. classes of extensions of G by A

$$\bar{\mathcal{E}}\underline{(G, A)} \longleftrightarrow H^2\underline{(G, A)} \; : \boxed{\text{As a set}}$$

$$H^2(G, A) = Z^2(G, A) / \sim$$

$Z^2(G, A)$: $f : G \times G \longrightarrow A$

s.t.

$$f(g_1, g_2) f(g_1 g_2, g_3) = f(g_2, g_3) f(g_1, g_2 g_3)$$

$$\sim : \quad \hat{f}(g_1, g_2) = f(g_1, g_2) \frac{t(g_1) t(g_2)}{t(g_1 g_2)}$$

for some $t : G \longrightarrow A$.

But $H^2(G,A)$ has a ~~t~~ group structure.

$\overline{\mathcal{E}}(G,A)$ can be intrinsically defined.

$$\mathcal{E}_1 : \quad 1 \to A \to \tilde{G}_1 \to G \to 1$$

$$\mathcal{E}_2 : \quad 1 \to A \to \tilde{G}_2 \to G \to 1$$

$$\begin{tikzcd}
 & 1 \arrow[r] & A \arrow[r] & \tilde{G}_{12} \arrow[r] & G \arrow[r] & 1 & \\
\mu(a_1,a_2) = a_1 a_2 & 1 \arrow[r] & A \times A \arrow[u, "\mu"] \arrow[r] & \hat{G}_{12} \arrow[r, two heads] & G \arrow[d, "\Delta"] & & g \arrow[d, mapsto] \\
\mathcal{E}_1 \times \mathcal{E}_1 & 1 \arrow[r] & A \times A \arrow[r, "\iota_1 \times \iota_2"] & \tilde{G}_1 \times \tilde{G}_2 \arrow[r, "\pi_1 \times \pi_2"] & G \times G \arrow[r] & 1 & (g,g)
\end{tikzcd}$$

$$\mathcal{E}_1 \cdot \mathcal{E}_2 = \mu_* \Delta^* (\mathcal{E}_1 \times \mathcal{E}_2)$$

$\mathcal{E}_1 \cdot \mathcal{E}_2$ should be an extension of G by A

Example 2 Extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_p \arrow[r] & \tilde{G} \arrow[r] & \mathbb{Z}_p \arrow[r] & 1 \end{tikzcd}$$

order $= p^2$

$$\Longrightarrow \quad \tilde{G} \cong \mathbb{Z}_p \times \mathbb{Z}_p$$

(in this case the extension is trivial)

or $\mathbb{Z}_{p^2}$: There are several inequivalent extensions.

But : $$H^2(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}_p \arrow[r, "2"] & \mathbb{Z}_{p^2} \arrow[r, "\pi"] & \mathbb{Z}_p \arrow[r] & 1 \end{tikzcd}$$

Conceptually different

So: Introduce notation:

$$\mathbb{Z}_p = \langle \sigma_1 \mid \sigma_1^p = 1 \rangle$$

$$\mathbb{Z}_{p^2} = \langle \alpha \mid \alpha^{p^2} = 1 \rangle$$

$$\mathbb{Z}_p = \langle \sigma_2 \mid \sigma_2^p = 1 \rangle$$

π is surjective

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_p \arrow[r, "\iota"] & \mathbb{Z}_{p^2} \arrow[r, "\pi"] & \mathbb{Z}_p \arrow[r] & 1 \\
 & \langle \sigma_1 \rangle & \langle \alpha \rangle & \langle \sigma_2 \rangle &
\end{tikzcd}$$

$$\iota(\sigma_1) = \alpha^x$$

$$\iota(1) = \iota(\sigma_1^p) = 1 \implies \alpha^{px} = 1$$

$$\implies x = 0 \mod p$$

ι injective so must be of form: $\boxed{\iota_k(\sigma_1) = \alpha^{kp}}$

where $k \in \mathbb{Z}_p^*$, $1 \leq k \leq p-1$

$(k, p) = 1$.

Similar: π must be of the form

$$\pi_r(\alpha) = \sigma_2^{\,r} \qquad 1 \leq r \leq p-1$$

$$r \in \mathbb{Z}_p^*$$

$\ker \pi_r \qquad \pi_r(\alpha^l) = \sigma_2^{\,lr} = 1$

$r \in \mathbb{Z}_p^* \qquad lr = 0 \mod p$

$\therefore\ l = 0 \mod p$

$$\ker \pi_r = \{1, \alpha^p, \alpha^{2p}, \dots, \alpha^{(p-1)p}\}$$

Check $= \operatorname{im} \iota_k$

Conc: We've found the most general possibilities for ι, π.

$\iota = \iota_k$ for some $k \in \mathbb{Z}_p^*$

$\pi = \pi_r$ for some $r \in \mathbb{Z}_p^*$

Compute a cocycle —

Choose a section :

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}_p \arrow[r, "\iota_K"] & \mathbb{Z}_{p^2} \arrow[r, "\pi_r"] & \mathbb{Z}_p \arrow[r] \arrow[l, dashed, bend left, "s"] \arrow[d, phantom, "\shortparallel"] & 1 \\
 & & & \langle \sigma_2 \rangle &
\end{tikzcd}$$

$$s(1) = 1$$

$$s(\sigma_2) = \alpha^x$$

$$\pi_r \circ s(\sigma_2) = \sigma_2$$

$$\therefore \quad \pi_r(\alpha^x) = \sigma_2^{rx} = \sigma_2$$

$$\therefore \quad rx = 1 \mod p$$

Choose simplest solution $\quad r \in \mathbb{Z}_p^*$

$$1 \leq r \leq p-1$$

$\exists!\quad 1 \leqslant r^* \leqslant p-1$

s.t. $\quad rr^* = 1 \mod p.$

$$s(\sigma_2) = \alpha^{r_*}$$

$$s(\sigma_2^2) = \alpha^{2r_*}$$

$$\vdots$$

$$s(\sigma_2^{p-1}) = \alpha^{(p-1)r_*}$$

$$\boxed{\begin{aligned} s(1) &= 1 \\ s(\sigma_2) &= \alpha^{r^*} \\ s(\sigma_2^2) &= \alpha^{2r^*} \\ &\vdots \\ s(\sigma_2^{p-1}) &= \alpha^{(p-1)r^*} \end{aligned}}$$

$$s(\sigma_2^{x}) \in \mathbb{Z}_{p^2} \overset{\shortparallel}{} \langle \alpha \rangle$$

Now $s(\sigma_2^p)$?

group homom. requires $s(\sigma_2^{p-1})s(\sigma_2)$

$$= \alpha^{(p-1)r^*}\alpha^{r^*} = \alpha^{pr^*}$$

but $\sigma_2^p = 1$ $\quad s(1) = 1$.

We're stuck: we can't choose s to be a group homomorphism.

Extension is non-split

Calculate the cocycle:

$$s(\sigma_2^x)\,s(\sigma_2^y)\,s(\sigma_2^{x+y})^{-1}$$

$$= \begin{cases} 1 & x+y \le p-1 \\ \alpha^{r^*_p} & x+y \ge p \end{cases}$$

$\therefore$

$$z_k(\sigma_1) = \alpha^k \qquad k \in \mathbb{Z}_p^*$$

$$\boxed{f_{k,r}(\sigma_2^x, \sigma_2^y) = \begin{cases} 1 & x+y \le p-1 \\ \sigma_1^{k^*r^*} & x+y \ge p \end{cases}}$$

$$\cap$$

$$A = \mathbb{Z}_p = \langle \sigma_1 \rangle$$

where $k^*k = 1 \mod p$

$\Rightarrow$ This is a cocycle.

Notes: Show how trying to trivialize with a coboundary won't work.

$$\begin{tikzcd}
 & & \langle \alpha \rangle \arrow[dd, "\varphi"] \arrow[dr, "\pi_{r_1}"] & & \\
1 \arrow[r] & \langle \sigma_1 \rangle \arrow[ur, "\iota_{k_1}"] \arrow[dr, "\iota_{k_2}"'] \arrow[r, phantom, "②"] & {} \arrow[r, phantom, "①"] & \langle \sigma_2 \rangle \arrow[r] & 1 \\
 & & \langle \alpha \rangle \arrow[ur, "\pi_{r_2}"'] & &
\end{tikzcd}$$

isom. of <u>extensions</u> is an iso of groups making The diagram commute.

$\varphi(\alpha) = \alpha^y$ $\quad$ y rel. prime to p

① commutes

$$\pi_{r_2} \circ \varphi(\alpha) = \pi_{r_1}(\alpha)$$

$$\Longrightarrow \quad r_2 y = r_1 \bmod p$$

② commutes: $\varphi\, \iota_{k_1}(\sigma_1) = \iota_{k_2}(\sigma_2)$

$$\Longrightarrow \quad k_1 p y = k_2 p \mod p^2$$

$$\Longleftrightarrow \quad k_1 y = k_2 \mod p.$$

$$r_2 y = r_1 \mod p$$

$$k_1 y = k_2 \mod p$$

and $y \in \mathbb{Z}_p^*$

$$\Longrightarrow \quad k_1 r_1 = k_2 r_2 \mod p$$

$[f_{k,r}]$ is determined by $k \cdot r$ or equiv. $k^* r^*$

$$[f_{k_1,r_1}] \cdot [f_{k_2,r_2}] \Longleftrightarrow \boxed{k_1^* r_1^*} + \boxed{k_2^* r_2^*} \mod p$$

$\mathbb{Z}_p$ ring $\quad \mathbb{Z}_p^* \subset \mathbb{Z}_p$ group of units.

$$H^2(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$$

$1 \le (kr)^* \le p-1 \quad$ $p-1$ extensions

with $\tilde{G} = \mathbb{Z}_{p^2}$ $\quad \tilde{G} = \mathbb{Z}_p \times \mathbb{Z}_p \; [f] = 1.$

Example 4 p prime $G \ni \vec{x}$

$$1 \to \mathbb{Z}_p \to \tilde{G} \longrightarrow \overbrace{\underbrace{\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p}_{k \text{ summands.}}}^{G} \to 1$$

Write additively

Cocycle condition

$$f : G \times G \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

$$f(\vec{x},\vec{y}) + f(\vec{x}+\vec{y},\vec{z})$$
$$= f(\vec{x},\vec{y}+\vec{z}) + f(\vec{y},\vec{z})$$

Satisfy that with

$$f(\vec{x},\vec{y}) = \underline{\underline{A_{ij}\, x_i\, y_j}} \leftarrow \boxed{\text{Just plug it in}}$$

$$A \in \mathrm{Mat}_{k \times k}(\mathbb{Z}_p)$$

Change by a coboundary

$$f(\vec{x},\vec{y}) \longrightarrow f(\vec{x},\vec{y}) + q(\vec{x}+\vec{y}) - q(\vec{x}) - q(\vec{y})$$

q ~ coboundary. $q : G = (\mathbb{Z}/p\mathbb{Z})^{\oplus k} \longrightarrow A = \mathbb{Z}/p\mathbb{Z}$

$$q(\vec{x}) = q_{ij}\, x_i x_j$$

$$A_{ij} \longrightarrow A_{ij} - (q_{ij} + q_{ji})$$

$\underline{p=2}$ put $A_{ij} = 0$ $i > j$ ←

$A_{ji} = 0$ or $\underline{1}$

gauge inv't.

$$A_{ii} \longrightarrow A_{ii} - \boxed{\underset{\overset{\shortparallel}{0}}{2}\, q_{ii}} = \underline{\underline{A_{ii}}}$$

$\frac{1}{2}k(k+1)$ $\boxed{\text{indpt}}$ cocycles.

$\mathbb{Z}_2^{\oplus \frac{1}{2}k(k+1)} \nearrow$

If p is an odd prime.

$$A_{ij} \longrightarrow A_{ij} - 2(q_{ij} + q_{ji})$$

So $(\mathbb{Z}/p\mathbb{Z})^{\frac{1}{2}k(k\underset{\uparrow}{-}1)}$

dim'l
space
of cohomology
classes.

Using topology
one computes
(Künneth theorem)

$$H^2(\mathbb{Z}_p^{\oplus k}, \mathbb{Z}_p) \cong \mathbb{Z}_p^{\frac{1}{2}k(k\overset{\downarrow}{+}1)}$$

But we just constructed p
nontrivial cocycles $H^2(\mathbb{Z}_p, \mathbb{Z}_p)$
not of the form $A_{ij}x_iy_j$

Example 5:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{G} \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1$$

$$H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

Distinct groups in the middle: 4 types

$$1 \to \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow 1$$

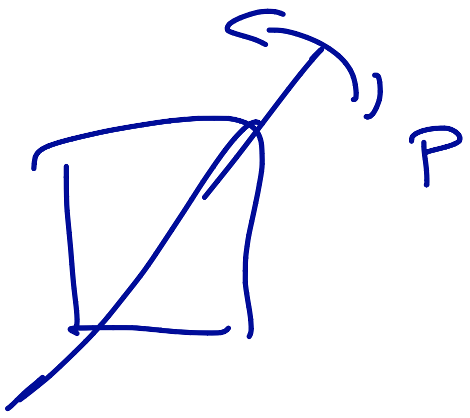
$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1$$

$$1 \to \mathbb{Z}_2 \longrightarrow Q \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1$$

$$1 \to \mathbb{Z}_2 \longrightarrow D_4 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 1$$

$$Q = \{\pm 1, \pm i\sigma^1, \pm i\sigma^2, \pm i\sigma^3\}$$

$$D_4 = \{1, R, R^2, R^3, P, PR, PR^2, PR^3\}$$



$R = R(\pi/2)$

Make an

Explicit choice of section $\Longrightarrow$ explicit bilinear form $A_{ij} x_i y_j$.

8 extensions

4 isom. of groups $\tilde{G}$.

Exercises:

1. Trivializable central extension is a direct product $\tilde{G} = A \times G$

Not true of split (noncentral) extensions

2. D_4 vs. Q.

3. Group Commutator Criterion.

$$\begin{tikzcd}
 & & \tilde{g}_1, \tilde{g}_2 & g_1 g_2 \arrow[l, bend right] & \\
1 \arrow[r] & A \arrow[r] & \tilde{G} \arrow[r, "\pi"'] & G \arrow[r] & 1
\end{tikzcd}$$

If $g_1, g_2 \in G$ are a commuting pair $g_1 g_2 = g_2 g_1$ and there are lifts $\tilde{g}_1, \tilde{g}_2$ s.t.

$$\pi(\tilde{g}_1) = g_1 \qquad \pi(\tilde{g}_2) = g_2$$

if $\underline{[\tilde{g}_1, \tilde{g}_2]} = z(a) \qquad a \neq 1$

then the extension is nontrivial.

$$H^2(\mathrm{Aut}(QM), U(1)) = ?$$

$$H^2(O(3), U(1)) \longleftarrow \boxed{\text{single Qbit}}$$

which coho? continuous.

$$H^*(BO(3), \mathbb{Z})$$

$\underline{\underline{\text{known}}}$

$$\langle w_1, w_2 \ldots p_1, \ldots \rangle$$

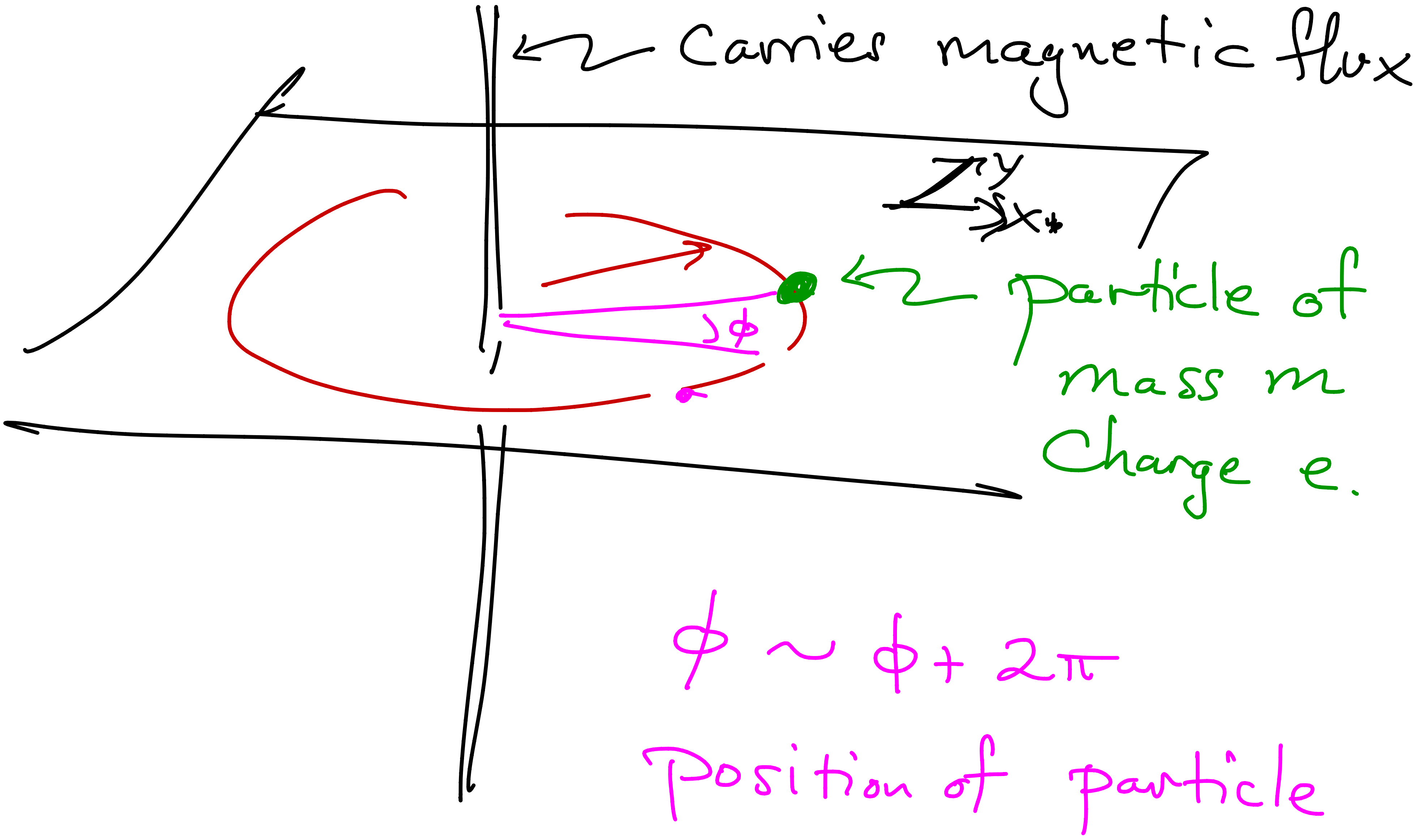
$\underline{w_1^2, w_2}$ $\underline{2\text{-torsion}}$.

Pure states $S^2 = \partial$ Bloch Sphere

F-S metric: round metric.

$$\mathrm{Aut}(QM) = \mathrm{Isom}(S^2, \text{round metric}) = O(3).$$

Quantum Mechanics Of A Charged Particle On a Ring Surrounding a Solenoid:



$$S = \int \frac{1}{2} m r^2 \dot{\phi}^2 \, dt$$

$$= \int \frac{1}{2} I \dot{\phi}^2 \, dt \qquad \text{w/out charge}$$

"

$$S = \int \frac{1}{2} I \dot{\phi}^2 + \int eA$$

gauge potential

worldline

As in Aharanov-Bohm.

In general $x^\mu(t)$ moving through a space with gauge field $A_\mu(x)$ particle has charge e mass m

metric on space

$$S' = \int \frac{1}{2} m \, g_{\mu\nu}(x(t)) \, \dot{x}^\mu(t) \, \dot{x}^\nu(t) \, dt$$

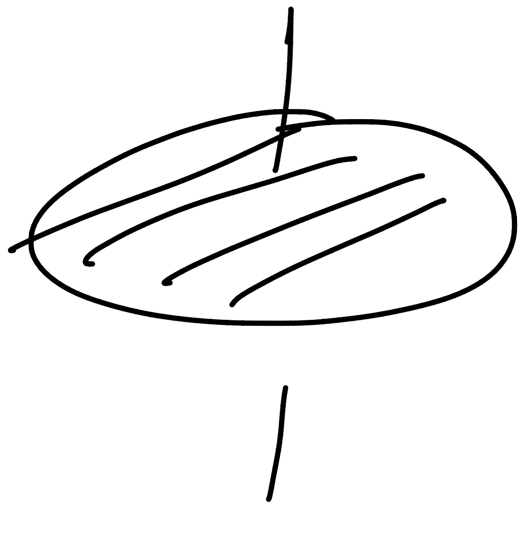
$$+ \underline{\int e A_\mu(x(t)) \, \dot{x}^\mu(t) \, dt}$$

A-B phase.

$$\int d[x(t)] \, e^{iS'}$$

$$S = \int \frac{1}{2} I \dot{\phi}^2 + \underline{\int e A_\mu(x(t)) \, \dot{x}^\mu(t) \, dt}$$

A 1-form $\quad F = dA$



$$\int F = B.$$

$$\Rightarrow A = \frac{B}{2\pi} d\phi$$

$$A_z = 0 \qquad A_r = 0 \qquad A_\phi = \frac{B}{2\pi}$$

$$\boxed{S = \int \frac{1}{2} I \dot{\phi}^2 dt + \frac{eB}{2\pi} \dot{\phi}\, dt}$$

inv't $\phi \to -\phi$

not

Classical Symmetries

Quantum Symmetries

Equations of motion: $\ddot{\phi} = 0.$

N.B.! B - term did not contribute.

$$\left.\int \underbrace{\frac{eB}{2\pi} \dot{\phi}}_{\text{total derivative}} dt \quad \right\} \quad \begin{array}{l}\text{example of} \\ \text{"}\theta\text{-term"} \\ \text{"topological term"}\end{array}$$

does not affect classical physics

<u>does</u> affect quantum physics

Classical eq's of motion $\ddot{\phi} = 0$

Symmetries $\phi \longrightarrow \phi + \alpha$ $\alpha \sim \alpha + 2\pi$

$R(\alpha) : \quad e^{i\phi(t)} \longrightarrow e^{i\phi(t)} e^{i\alpha}$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

P: $\phi \to -\phi$ reflection in xy plane

Symmetry of eqs of motion by not of the top. term.

$R(\alpha)\,R(\beta) = R(\alpha+\beta)$ SO(2)

$P^2 = 1$

$P R(\alpha) P = R(-\alpha)$

$\boxed{O(2)}$ Classical Symmetry Group

Compute Hamiltonian :

Canonical momentum conj. to ϕ

$$L = \frac{\delta S}{\delta \dot{\phi}} = I\dot{\phi} + \frac{eB}{2\pi}$$

$$\int H\,dt = \int L\dot{\phi}\,dt - S'$$

$$= \int \frac{1}{2I}\left(L - \frac{eB}{2\pi}\right)^2$$

Quant: $L = -i\hbar\frac{\partial}{\partial\phi}$ acting on L^2 - wavefunctions of ϕ

$$L = -i\hbar\frac{\partial}{\partial\phi} + \boxed{\underline{\underline{\cancel{\text{Const.}}}}} \to 0.$$

$$H_{\mathcal{B}} = \frac{\hbar^2}{2I}\left(-i\frac{\partial}{\partial\phi} - \mathcal{B}\right)^2$$

$$\mathcal{B} = \frac{eB}{2\pi\hbar}$$

Diagonalize $\mathcal{H} = L^2(S^1)$

Complete set: $\overset{\text{ON}}{\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}}$ $m \in \mathbb{Z}$

also eigenvectors of H

$$H_B \psi_m = E_m \psi_m$$

$$E_m = \frac{\hbar^2}{2I}(m - B)^2$$

1- parameter family of Hamiltonians

$$\boxed{B \to B + c}$$

does not change the family.

Remarks:

$$\underline{B^{\text{eff}} = B + c}$$ $$\boxed{c = 0}$$

1. Topological term matters because the observable spectrum is <u>shifted</u> by $B = eB/2\pi\hbar$

2. Important that $m \in \mathbb{Z}$ because $\phi \sim \phi + 2\pi$

Action $S[\phi(t)]$ also makes

Sense for $\phi(t) \in \mathbb{R}$

without the identification $\phi \sim \phi + 2\pi$

But then m would not be quantized and we could shift away $\mathcal{B}$ (or $\mathcal{B}^{\text{eff}}$).

Topological term would have no effect in the quantum theory as well.

Need to have some topology.

$$\operatorname{Tr}\left[\frac{1}{H_{\mathcal{B}} - z}\right] = R(z)$$

Very useful. $\frac{1}{H_{\mathcal{B}} - z}$ Resolvent

Next time:

After some further remarks
We'll look at the quantum symmetries.